

Quantum Lower Bound for Graph Collision Implies Lower Bound for Triangle Detection

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Abstract. We show that an improvement to the best known quantum lower bound for GRAPH-COLLISION problem implies an improvement to the best known lower bound for TRIANGLE problem in the quantum query complexity model. In GRAPH-COLLISION we are given free access to a graph (V, E) and access to a function $f : V \rightarrow \{0, 1\}$ as a black box. We are asked to determine if there exist $(u, v) \in E$, such that $f(u) = f(v) = 1$. In TRIANGLE we have a black box access to an adjacency matrix of a graph and we have to determine if the graph contains a triangle. For both of these problems the known lower bounds are trivial ($\Omega(\sqrt{n})$ and $\Omega(n)$, respectively) and there is no known matching upper bound.

1 Introduction

By $Q(f)$ we denote the bounded-error quantum query complexity of a function f . We consider the quantum query complexity for some graph problems.

Definition 1. In TRIANGLE problem it is asked whether an n -vertex graph $G = (V, E)$ contains a triangle, i.e. a complete subgraph on three vertices. The adjacency matrix of the graph is given in a black box which can be queried by asking if $(x, y) \in E$.

Recently there have been several improvements in the algorithms for the TRIANGLE problem in the quantum black box model. The problem was first considered by Buhrman et al. in 2005 [4] who gave an $O(n + \sqrt{nm})$ algorithm where n is the number of vertices and m – the number of edges. Later in 2007 Magniez et al. gave an $\tilde{O}(n^{13/10})$ algorithm based on quantum walks. Introducing a novel concept – learning graphs, and using a new technique in 2012 Belovs [3] was able to reduce the complexity to $O(n^{35/27})$. In 2013 Lee et al. [8] using a more refined learning graph approach reduced the complexity to $\tilde{O}(n^{9/7})$. Currently the best known algorithm is by Le Gall who exhibited a quantum algorithm which solves the TRIANGLE problem with query complexity $\tilde{O}(n^{5/4})$ [5]. Classically the query complexity of TRIANGLE is $\Theta(n^2)$; however, it is an

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open question whether TRIANGLE can be computed in time better than $O(n^\omega)$ where ω is the matrix multiplication constant.

Definition 2. In GRAPH-COLLISION_G problem a known n -vertex undirected graph $G = (V, E)$ is given and a coloring function $f : V \rightarrow \{0, 1\}$ whose values can be obtained by querying the black box for the value of $f(x)$ of a given $x \in V$. We say that a vertex $x \in V$ is marked iff $f(x) = 1$. The value of the GRAPH-COLLISION_G instance is 1 iff there exists an edge whose both vertices are marked, i.e. $\exists (x, y) \in E$ $f(x) = f(y) = 1$.

By $Q(\text{GRAPH-COLLISION})$ we mean the complexity of solving GRAPH-COLLISION_G for the hardest n -vertex graph G .

There has been an increased interest in the quantum query complexity of the GRAPH-COLLISION problem, mainly because algorithms for solving GRAPH-COLLISION are used as a subroutine in algorithms for the TRIANGLE problem [9] and Boolean matrix multiplication [7].

The best known quantum algorithm for GRAPH-COLLISION for an arbitrary n -vertex graph has complexity $O(n^{2/3})$ [9]. However, for some graph classes there are algorithms with complexity $O(\sqrt{n})$ [1, 2, 6, 7]. It is an open question whether for every n -vertex graph G GRAPH-COLLISION_G can be solved with $O(\sqrt{n})$ queries.

Contrary to the improvements in the algorithms for these two problems, the best known lower bounds for $Q(\text{GRAPH-COLLISION})$ and $Q(\text{TRIANGLE})$ are still the trivial $\Omega(\sqrt{n})$ and $\Omega(n)$ respectively, which follow from the reduction to search problem. Nonetheless these lower bounds seem hard to improve with the current techniques.

As mentioned before, algorithms for GRAPH-COLLISION have been used as a subroutine for constructing algorithms for the TRIANGLE problem, therefore an improved algorithm for GRAPH-COLLISION would result in an improved algorithm for TRIANGLE. In this paper we show a reduction in the opposite direction—that an improvement in the lower bound on $Q(\text{GRAPH-COLLISION})$ would imply an improvement in the lower bound on $Q(\text{TRIANGLE})$.

2 Result

Theorem 1. *If there is a graph $G = (V, E)$ with n vertices such that GRAPH-COLLISION_G has quantum query complexity t then TRIANGLE problem has quantum query complexity at least $\Omega(t\sqrt{n})$.*

Proof. We show how to transform the graph G into a graph G' with $3n$ vertices so that it is hard to decide if G' contains a triangle. More precisely, we construct the graph G' in such a way that solving the TRIANGLE problem on G' is equivalent to solving OR function from the results of n independent instances of GRAPH-COLLISION_G.

First, we want to get rid of any triangles in G , therefore we transform G into an equivalent bipartite graph $G_2 = (V_2, E_2)$ with $2n$ vertices by setting

$V_2 = \{v_1, v_2 \mid v \in V\}$ and $E_2 = \{(x_1, y_2) \mid (x, y) \in E\}$. The graph G_2 is equivalent to G in the following sense—if we mark the vertices v_1 and v_2 in G_2 for every marked vertex v in G , then G_2 has a collision iff G has a collision. However, the graph G_2 does not contain any triangle (since it is bipartite).

Next, we add n isolated vertices z_1, \dots, z_n to G_2 thereby obtaining a graph G' . Let $f_1, \dots, f_n : V \rightarrow \{0, 1\}$ be the colorings from n independent GRAPH-COLLISION_G instances. We add the edges (z_i, v_1) and (z_i, v_2) to G' iff $v \in V$ is marked by the respective coloring, i.e., iff $f_i(v) = 1$.

See Fig. 1 for an example.

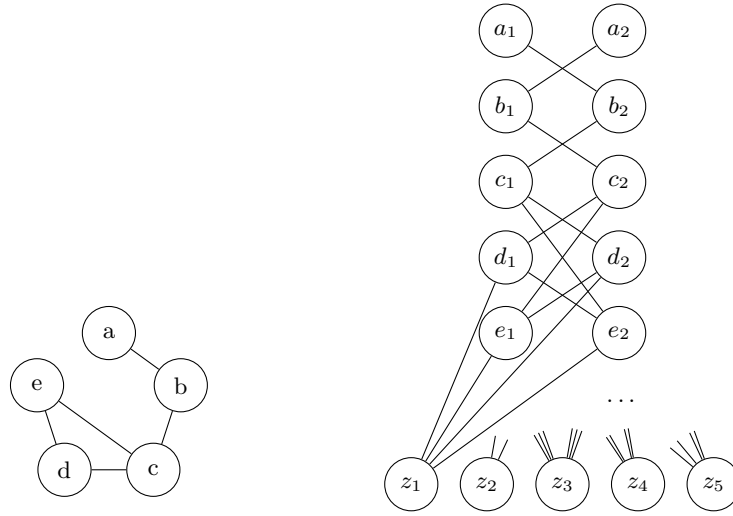


Fig. 1. Graph G and the resulting graph G'

The only possible triangles in the graph G' can be of the form $\{z_i, v_1, w_2\}$ for some $i \in \{1, \dots, n\}$ and $v, w \in V$. Moreover, there is a triangle $\{z_i, v_1, w_2\}$ iff f_i is such coloring that G has a collision (v, w) , i.e., iff $f_i(v) = f_i(w) = 1$. Therefore detecting a triangle in G' is essentially calculating OR function from the results of n instances of GRAPH-COLLISION_G .

We now use the fact that OR function requires $\Omega(\sqrt{n})$ queries, the assumption that GRAPH-COLLISION_G requires t queries and the Theorem 1.5. from [10]:

Theorem 2. Let $f : \{0, 1\}^n \rightarrow \{0, 1\}$ and $g : \{0, 1\}^m \rightarrow \{0, 1\}$. Then

$$Q(f \bullet g) = \Theta(Q(f)Q(g)),$$

where $(f \bullet g)(x) = f(g(x_1, \dots, x_m), \dots, g(x_{(n-1)m+1}, \dots, x_{nm}))$.

Setting $f = OR$ and $g = \text{GRAPH-COLLISION}_G$ gives the desired bound.

As the next corollary shows, a better lower bound on GRAPH-COLLISION implies a better lower bound on the TRIANGLE problem.

Corollary 1. *If $Q_2(\text{GRAPH-COLLISION}) = \omega(\sqrt{n})$ then $Q_2(\text{TRIANGLE}) = \omega(n)$.*

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